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The adjoint weighted equation for steady advection in a compressible fluid

Assad A. Oberai[1], Paul E. Barbone[2] and Isaac Harari[3,*,†]

[1]*Department of Mechanical, Aerospace, and Nuclear Engineering, Rensselaer Polytechnic Institute, Troy, NY 12180, U.S.A.*
[2]*Department of Aerospace and Mechanical Engineering, Boston University, Boston, MA 02215, U.S.A.*
[3]*Department of Solid Mechanics, Materials, and Systems, Tel Aviv University, 69978 Ramat Aviv, Israel*

SUMMARY

An alternative variational framework suitable for pure advection is obtained by discarding the Galerkin part of stabilized methods. The resulting scheme is similar to the least-squares approach, but with the adjoint operator in the weighting slot. This formulation is not restricted to solenoidal (i.e. divergence free) velocities. Initial numerical results for such problems show that the method is promising. Copyright © 2007 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Two common approaches to computing transport phenomena are based on least-squares and stabilized formulations. The former are robust and stable [1], but may require sophisticated techniques to retain desired accuracy in some cases [2]. On the other hand, the performance of stabilized methods is determined by the choice of the mesh-dependent stabilization parameters that are inherent in their formulation [3].

The adjoint weighted equation (AWE) formulation, which is an alternative variational framework suitable for pure advection, may be viewed as a combination of the two. Work on the related nearly optimal Petrov–Galerkin method [4] prompted the observation motivating this concept, that

*Correspondence to: Isaac Harari, Department of Solid Mechanics, Materials, and Systems, Tel Aviv University, 69978 Ramat Aviv, Israel.
†E-mail: harari@eng.tau.ac.il

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in the advective limit certain stabilized methods, including the streamline upwind Petrov–Galerkin method [5], perform well for arbitrarily large values of the stabilization parameter, so that the Galerkin part may be discarded. The resulting scheme is similar to the least-squares approach, but employs the adjoint operator in the weighting slot.

The hyperbolic boundary-value problem for scalar advective transport, in which the velocity field need not be solenoidal (i.e. divergence free), is formulated in Section 2, and common numerical approaches are reviewed. The AWE formulation is introduced in Section 3, and several of its features are discussed. Numerical tests described in Section 4 show promising results. Section 5 offers conclusions.

2. BOUNDARY-VALUE PROBLEM

Let $\Omega \subset \mathbb{R}^d$ be a d-dimensional, open, bounded region with smooth boundary Γ. The outward unit vector, normal to Γ, is denoted $\mathbf{n}$. The advective velocity field $\mathbf{a}$ is given. The velocity field is often assumed to be solenoidal ($\nabla \cdot \mathbf{a} = 0$), describing an incompressible flow. This restriction is *not* imposed in the following. Thus, the formulation presented here is applicable to both compressible and incompressible flow fields.

The boundary Γ is partitioned according to the characteristics into an inflow boundary

$$\Gamma^- = \{\mathbf{x} \in \Gamma | \mathbf{a}(\mathbf{x}) \cdot \mathbf{n}(\mathbf{x}) < 0\} \quad (1)$$

meas$(\Gamma^-) > 0$, and an outflow boundary, $\Gamma^+ = \Gamma \backslash \Gamma^-$; see Figure 1.

2.1. Scalar advective transport

Consider the hyperbolic boundary-value problem: find $u : \overline{\Omega} \to \mathbb{R}$ that satisfies

$$\nabla \cdot (\mathbf{a}u) = f \quad \text{in } \Omega \quad (2)$$

$$u = g \quad \text{on } \Gamma^- \quad (3)$$

Here, $f : \Omega \to \mathbb{R}$ and $g : \Gamma^- \to \mathbb{R}$ are prescribed.

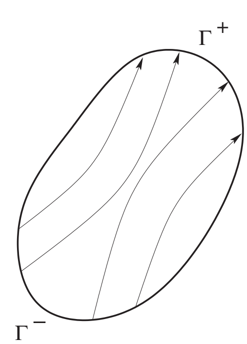

Figure 1. Boundary partition.

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2.2. *Weak form and Galerkin approximation*

The variational formulation is stated in terms of the sets of functions

$$\mathscr{S} = \{u \in H^1(\Omega) | u = g \text{ on } \Gamma^-\} \tag{4}$$

$$\mathscr{V} = \{w \in H^1(\Omega) | w = 0 \text{ on } \Gamma^-\} \tag{5}$$

The standard weak form of the scalar advective problem (2)–(3) is: find $u \in \mathscr{S}$ such that

$$(w, \nabla \cdot (\mathbf{a}u)) = (w, f) \quad \forall w \in \mathscr{V} \tag{6}$$

Here, $(\cdot, \cdot)$ is the $L_2(\Omega)$ inner product.

In order to examine the stability of the continuous case (6), consider the left-hand side operator (see, e.g. [6])

$$\begin{aligned}(w, \nabla \cdot (\mathbf{a}w)) &= (w, \mathbf{a} \cdot \mathbf{n}w)_\Gamma - (\nabla w, \mathbf{a}w)\\ &= (w, \mathbf{a} \cdot \mathbf{n}w)_\Gamma - \tfrac{1}{2}(\nabla(w^2), \mathbf{a})\\ &= \tfrac{1}{2}(w, \mathbf{a} \cdot \mathbf{n}w)_\Gamma + \tfrac{1}{2}(w, (\nabla \cdot \mathbf{a})w)\end{aligned} \tag{7}$$

The first term is positive, but related to a weak norm that does not provide control on the variation of the function (see, e.g. [7]). The second term is indefinite unless the velocity field is solenoidal. Consequently, small variations in the data can lead to large variations in the solution.

The standard Galerkin finite element approximation is constructed by replacing the functions in the weak form (6) with finite-dimensional counterparts, typically containing continuous piecewise polynomials. The support of these functions is defined by a mesh partition of the domain Ω into non-overlapping regions (element domains), with mesh parameter h. The space of approximate weighting functions is denoted by $\mathscr{V}^h \subset \mathscr{V}$. The set of approximate trial solutions is denoted by $\mathscr{S}^h$. The standard finite element method is: find $u^h \in \mathscr{S}^h$ such that

$$(w^h, \nabla \cdot (\mathbf{a}u^h)) = (w^h, f) \quad \forall w^h \in \mathscr{V}^h \tag{8}$$

This method inherits the instability of the continuous problem and exhibits spurious oscillations in computation.

2.3. *Stabilized methods*

Stabilized finite element methods can alleviate this difficulty (see, e.g. a recent review [3]). The streamline upwind Petrov–Galerkin method [5] for the scalar advective problem (2)–(3) is: find $u^h \in \mathscr{S}^h$ such that

$$(w^h + \tau\mathbf{a} \cdot \nabla w^h, \nabla \cdot (\mathbf{a}u^h)) = (w^h + \tau\mathbf{a} \cdot \nabla w^h, f) \quad \forall w^h \in \mathscr{V}^h \tag{9}$$

The terms added to the standard formulation preserve consistency and enhance stability. Suitable definition of the mesh-dependent stabilization parameter, τ, is a subject of ongoing investigations [8].

2.4. *Least-squares formulation*

A standard least-squares formulation is robust and stable, free of the deficiencies of the weak form (6) [1], and independent of a mesh-dependent parameter, τ. A least-squares principle for the advective problem (2)–(3) is to seek a minimizer of the quadratic functional

$$J(u) = \|\nabla \cdot (\mathbf{a}u) - f\|^2 \tag{10}$$

in terms of the $L_2(\Omega)$ norm.

The necessary minimum condition is to seek $u \in \mathscr{S}$ such that

$$(\nabla \cdot (\mathbf{a}w), \nabla \cdot (\mathbf{a}u)) = (\nabla \cdot (\mathbf{a}w), f) \quad \forall w \in \mathscr{V} \tag{11}$$

Setting $u = w$ in the first term of the least-squares formulation (11) gives

$$\begin{aligned}(\nabla \cdot (\mathbf{a}w), \nabla \cdot (\mathbf{a}w)) &= \|\nabla \cdot (\mathbf{a}w)\|^2 \\ &= \|\mathbf{a} \cdot \nabla w + (\nabla \cdot \mathbf{a})w\|^2 \\ &\geqslant \frac{1}{1+\varepsilon}\|\mathbf{a} \cdot \nabla w\|^2 - \frac{\|\nabla \cdot \mathbf{a}\|^2}{\varepsilon}\|w\|^2 \quad \forall \varepsilon > 0\end{aligned} \tag{12}$$

The last line follows from the inequality

$$\|u + v\|^2 \geqslant \frac{1}{1+\varepsilon}\|u\|^2 - \frac{1}{\varepsilon}\|v\|^2 \quad \forall \varepsilon > 0 \tag{13}$$

See, e.g. [9]. The first term in (12) is non-negative. Stability depends on the data. Stability clearly holds for a solenoidal velocity field and remains so for sufficiently small $\nabla \cdot \mathbf{a}$.

Least-squares formulations often lead to overly dissipative solutions. Sophisticated techniques such as negative norms provide desired accuracy in some cases [2], but may entail implementational difficulties, for example in the specification of boundary conditions.

3. ADJOINT WEIGHTED EQUATION

Examining Petrov–Galerkin methods with the goal of obtaining optimal weighting functions [4] leads to the observation that the streamline upwind Petrov–Galerkin method for the advective problem performs well for arbitrarily large values of the stabilization parameter, yielding a formulation which retains only the stabilization terms.

3.1. *Formulation*

This alternative variational framework, the AWE formulation is: find $u \in \mathscr{S}$ such that

$$(\mathbf{a} \cdot \nabla w, \nabla \cdot (\mathbf{a}u)) = (\mathbf{a} \cdot \nabla w, f) \quad \forall w \in \mathscr{V} \tag{14}$$

This formulation, motivated by the streamline upwind Petrov–Galerkin method, is similar to one derived by least squares, but weighted by the *adjoint* operator. The least-squares (11) and AWE (14) formulations coincide for solenoidal velocity fields.

3.2. Euler–Lagrange equations

The Euler–Lagrange equations emanating from the AWE formulation (14) are obtained by integration by parts, assuming sufficient smoothness of the functions involved. Doing so yields the following boundary-value problem:

$$\nabla \cdot (\mathbf{a}\nabla \cdot (\mathbf{a}u)) = \nabla \cdot (\mathbf{a}f) \quad \text{in } \Omega \tag{15}$$

$$\mathbf{a} \cdot \mathbf{n}\nabla \cdot (\mathbf{a}u) = \mathbf{a} \cdot \mathbf{n}f \quad \text{on } \Gamma \tag{16}$$

$$u = g \quad \text{on } \Gamma^- \tag{17}$$

3.3. Stability

Setting $u = w$ in the first term of the AWE formulation (14) gives

$$\begin{aligned}(\mathbf{a} \cdot \nabla w, \nabla \cdot (\mathbf{a}w)) &= \|\mathbf{a} \cdot \nabla w\|^2 + (\mathbf{a} \cdot \nabla w, (\nabla \cdot \mathbf{a})w) \\ &\geqslant \|\mathbf{a} \cdot \nabla w\|^2 - \varepsilon\|\mathbf{a} \cdot \nabla w\|^2 - \frac{1}{4\varepsilon}\|(\nabla \cdot \mathbf{a})w\|^2 \\ &\geqslant (1-\varepsilon)\|\mathbf{a} \cdot \nabla w\|^2 - \frac{\|\nabla \cdot \mathbf{a}\|^2}{4\varepsilon}\|w\|^2 \quad \forall \varepsilon > 0\end{aligned} \tag{18}$$

The second line follows from the inequality

$$(u, v) \geqslant -\varepsilon\|u\|^2 - \frac{1}{4\varepsilon}\|v\|^2 \quad \forall \varepsilon > 0 \tag{19}$$

See, e.g. [9]. Taking $\varepsilon < 1$, the first term in (18) is non-negative. As in the least-squares formulation (11), stability depends on the data. Stability clearly holds for a solenoidal velocity field and remains so for sufficiently small $\nabla \cdot \mathbf{a}$.

3.4. Discretization

The Galerkin finite element approximation of the AWE formulation (14) is: find $u^h \in \mathscr{S}^h$ such that

$$(\mathbf{a} \cdot \nabla w^h, \nabla \cdot (\mathbf{a}u^h)) = (\mathbf{a} \cdot \nabla w^h, f) \quad \forall w^h \in \mathscr{V}^h \tag{20}$$

4. COMPUTATIONAL EXAMPLES

A series of computations examines the numerical performance of the proposed method in comparison to the least-squares formulation. These employ structured and distorted meshes of four-noded quadrilaterals, for two scalar advection problems in square domains, with velocity fields that are not solenoidal.

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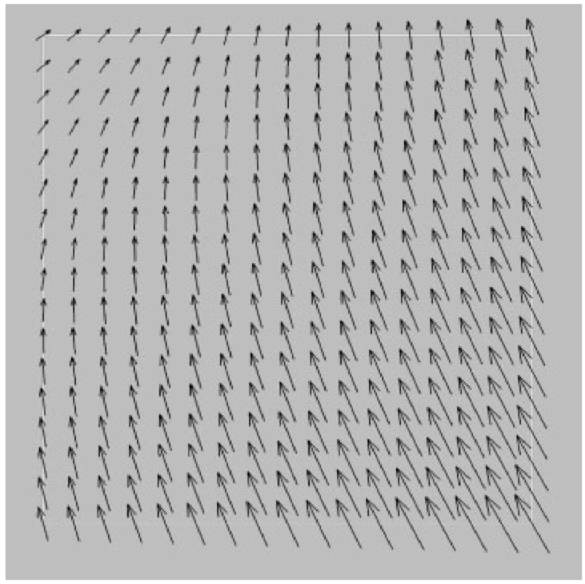

Figure 2. Linear velocity field (21) for the exponentially varying solution.

4.1. Exponentially varying solution

As the first problem we consider a case for which the analytical solution is known. The velocity is given by (see Figure 2)

$$\mathbf{a}=\begin{Bmatrix} (y-x)/2-0.2 \\ (x-y)/2+0.7 \end{Bmatrix} \tag{21}$$

Note that the velocity field is linear and is exactly represented by bilinear finite element functions. Further, $\nabla\cdot\mathbf{a}=-1$ and hence the AWE solution is expected to be different from the LS solution. The domain is a unit square. The inflow boundary is comprised of the edges $y=0$, $x=1$, and the upper part of $x=0$, $0.4<y<1$. There is no forcing term ($f=0$), and the boundary conditions are specified on the inflow boundary so that the exact solution is

$$u=\exp(2(x+y)) \tag{22}$$

The domain is discretized by two sets of seven increasingly refined meshes composed of bilinear quadrilateral finite elements. The first set consists of nested uniform meshes from 4×4 to 256×256 elements, with the element edges halved from one level of refinement to the next. In the second set the same refinement levels are used but the elements are randomly distorted by an average of 10% of the edge length. One such mesh corresponding to 16 elements in each direction is shown in Figure 3. Once the finite element solution is evaluated, its error is calculated using Gaussian quadrature. To check the accuracy of this computation, results with 4-point (per direction) and 5-point rules were compared and no significant difference was found.

An example of the solutions on the 4×4 distorted mesh along $x=0.5$ is shown in Figure 4. Errors for AWE and LS solutions on uniform meshes are reported in Figure 5, measured in the L_2

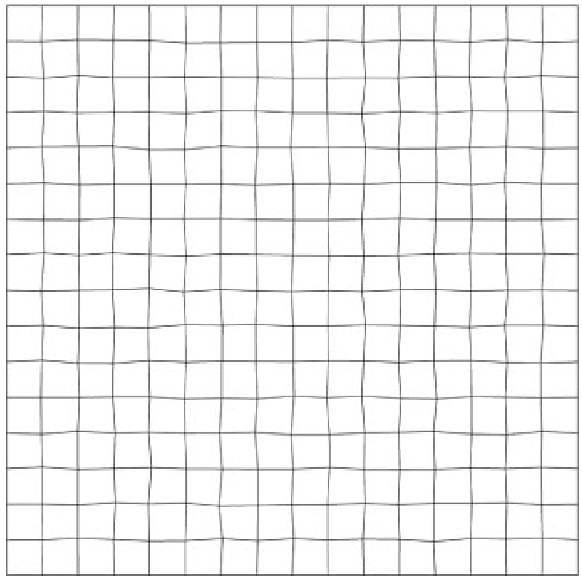

Figure 3. A randomly distorted 16×16 finite element mesh for the problem with the exponentially varying solution.

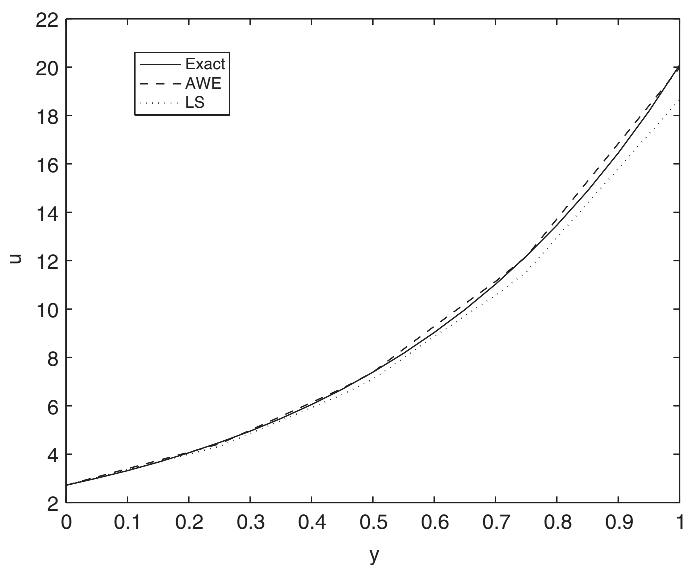

Figure 4. Solutions on the 4×4 distorted mesh along $x = 0.5$, for the problem with the exponentially varying solution.

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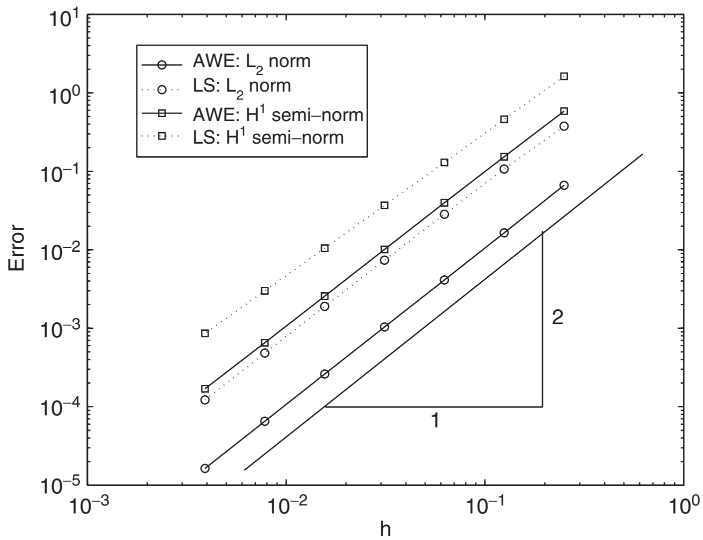

Figure 5. Error in the uniform meshes for the problem with the exponentially varying solution.

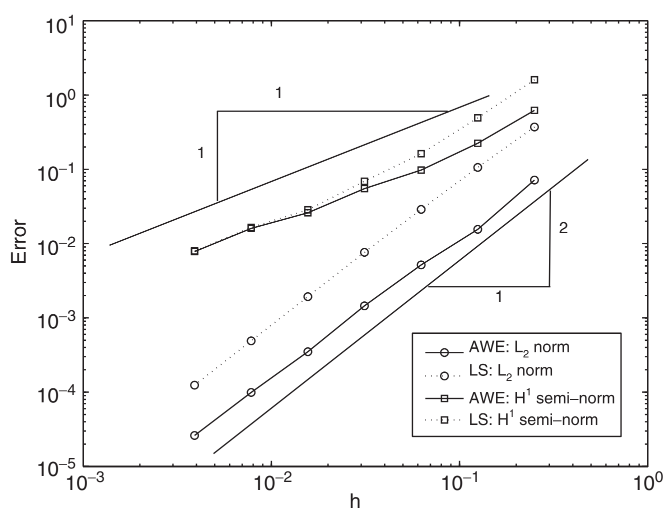

Figure 6. Error in the distorted meshes for the problem with the exponentially varying solution.

norm and H^1 seminorm. All curves have a slope close to 2, which points to superconvergence in the H^1 seminorm on uniform meshes. We also observe that while the rate of convergence for the AWE and LS solutions is the same, the former appear to have a smaller error (about 7–8 times smaller) for all meshes.

Figure 6 shows error in the solution with distorted meshes, once again measured in the L_2 norm and H^1 seminorm. We observe that superconvergence in the H^1 seminorm is lost, however, the

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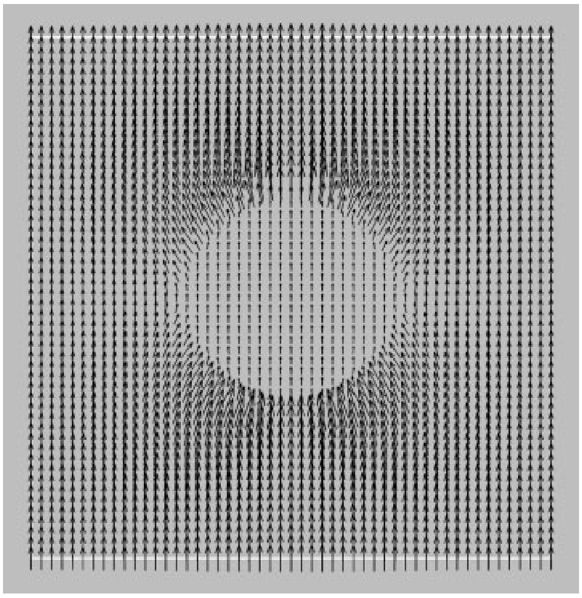

Figure 7. Velocity field for the circular plateau problem.

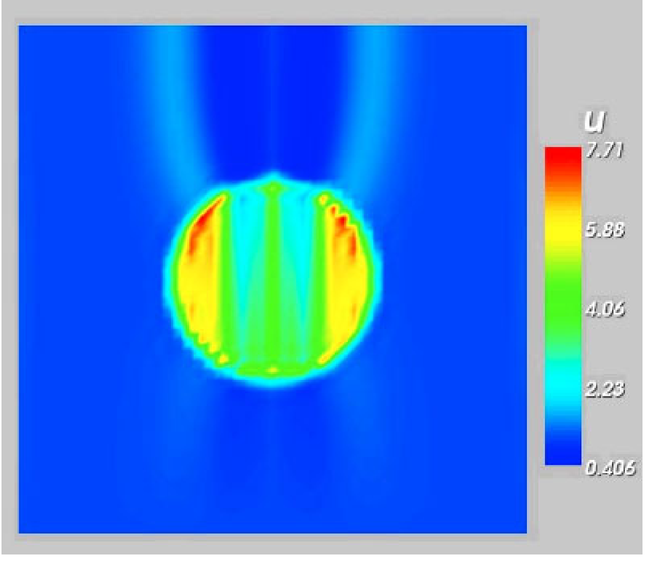

Figure 8. AWE solution for the circular plateau problem.

rate of convergence is optimal. Further, the AWE solution is more accurate than the LS solution for coarse meshes. The convergence rates in the L_2 norm are also optimal. Here, the superiority of AWE over LS is apparent throughout the range of refinement.

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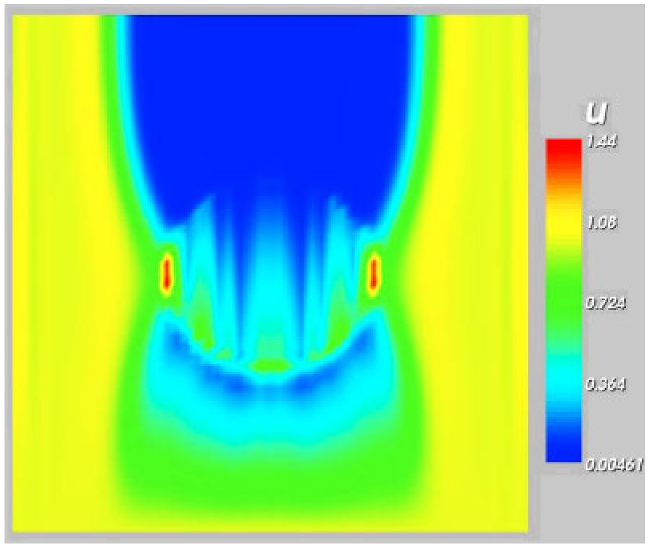

Figure 9. Least-squares solution for the circular plateau problem.

4.2. *Circular plateau*

The second problem we consider is more challenging. Here, the exact solution is in the form of a sharp circular plateau of diameter 0.4 in the centre of a unit square. The magnitude of the advected scalar u varies from unity in the background to 5 on the plateau.

The velocity field for this problem, with no forcing term, was computed on a fine mesh with u assigned to the desired profile. This velocity field, shown in Figure 7, is also not solenoidal.

The advection problem was solved on the unit square with 50 uniform bilinear finite elements in each direction. The $y=0$ edge was treated as inflow boundary and the solution was set to unity here. The AWE solution is shown in Figure 8. We observe that the location and the shape of the plateau are captured accurately although there are some spurious oscillations in the solution. The LS solution for this problem is shown in Figure 9. We observe that the solution is overly diffuse and the shape of the circular plateau is not captured accurately. Once again the superiority of the AWE solution over the LS solution is clear.

5. CONCLUSIONS

The adjoint weighted equation is an alternative variational framework for computing pure advection transport, motivated by the study of Petrov–Galerkin methods. This scheme is similar to the conventional least-squares approach, but employs the adjoint operator in the weighting slot. The concept may also be viewed as a stabilized method in which the Galerkin part was discarded, precluding the need for mesh-dependent stabilization parameters.

The adjoint weighted equation is not restricted to solenoidal (i.e. divergence free) velocities. Preliminary analysis indicates that the adjoint weighted equation shares the robustness of the

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least-squares approach, yet in computational tests provides superior numerical performance on the problems considered.

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